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VARIABLES AND LIMITS.

BY DUNHAM JACKSON.

The serious student of pure mathematics often has occasion for the uncomfortable belief that if his science is regarded with any interest at all, it is because it has ascribed to it certain qualities of doubtful merit which it does not possess. It is supposed to be in some way outside the bounds of normal human thought, and to deal with processes unintelligible to rational beings. When the mathematician is not making use of the fourth dimension to prove theorems that aren't so, he is lightly tossing infinity from one side of an equation to the other and pretending to be edified thereby. The proudest monument of mathematics is the infinitesimal calculus; and what is an infinitesimal? It is an infinitely small quantity, a quantity which is simultaneously zero and different from zero, at the same time there and not there, a thing essentially self-contradictory and intrinsically impossible. The one thing that you need, in order to be a mathematician, is an unhesitating readiness to identify truth with falsehood.

Your fellow-citizen not only comes to you with this opinion of your integrity; he resents it bitterly if you try to modify his views. He thinks you are denying everything in mathematics that is worth while. If you can't rise above the limitations of logical consistency that embarrass other people, what are you good for? You are no better than the man who called himself an astronomer, but couldn't see the moon unless it was above the horizon. A fellow-student of mine in freshman trigonometry fought valiantly against the professor's contention that an angle of 90° didn't have a tangent in quite the same sense as an ordinary angle, and when he did yield to superior authority, it was with the reservation that he thought it was awfully hard on 90°.

Why is it that mathematics is so generally misjudged, even among educated men? One important reason, I think, is this, that one of the most fundamental of mathematical notions remains in the background throughout the elementary study of the subject. It is present, but plays so little part that it is scarcely recognized. I mean the notion of *function,* the association of two variable quantities so that a value of one corresponds to a value of the other.

The concept is so abstract that it can be really appreciated only after long familiarity with it, more familiarity than the student of a moderate amount of mathematics ever gets. And yet it is of hardly less universal importance than the idea of number itself. Most of us acquired the latter notion at such an early age that we have almost forgotten what a long and painful process it was. It was when we first tried to think of variable quantities that we were conscious of a blank in our thoughts where a meaning ought to be. How could you tell by looking at a quantity whether it was variable or not, and what made it vary, and if its very essence was uncertainty, how could you do anything with it? It is at this point that the idea of *function* begins to be a blessing. For it turns out that when you are really doing mathematics you almost never have to think of a variable by itself; you have two variables in sight at the same time, and what you are concerned with is a relation between them, a correspondence of particular values of one variable to particular values of the other. There may be more than two variables, but there are always at least two. It is not easy to say in general terms how this correspondence comes to establish itself in your mind, nor is it necessary to do so. Very few of us, if any, could give an intelligible account of what the ordinary numbers of arithmetic mean to us, but we operate with them confidently enough, because we are familiar with them and know their ways. In the same way, we find that the notion of correspondence between variables stands by us faithfully in one crisis after another. We don't always know where it will come from, but we can be sure that it will be there. Let me give some examples of what I mean.

In elementary algebra we learn to write the equation

$$x^2 - 3x + 2 = (x - 1)(x - 2).$$

The left-hand member of this identity indicates a variable quantity which depends for its value on another variable quantity x. To any particular numerical value of x corresponds a perfectly definite number $x^2 - 3x + 2$, found by subjecting the number x to the simple arithmetical processes suggested by the formula. We say that $x^2 - 3x + 2$ is a *function* of x. The right-hand member of the identity gives another rule for calculating a second number whenever a value of x is given. The identity states that for any one value of x the associated number given by the first formula will always be the same as the associated number given by the second. These observations, trivial as they are, mean that the algebraic equation expresses the identity of two functions.

Suppose that we undertake to solve the equation $x^2 - 3x + 2 = 0$. It may seem that we are only solving a puzzle, moving the counters in accordance with the rules of the game. But we can attach more meaning to the problem. We have given a function of x; to different values of x correspond different values of the function. With what values of x is the value zero associated? Every problem involving the solution of an equation is a problem involving a functional relation.

The circumference of a circle is $2\pi r$. With a value of the radius is associated a value of the circumference. One is a function of the other. In a given circle, a regular inscribed polygon of n sides has a certain definite perimeter, depending on the value of n. Ordinarily, we do not think particularly of the value of the perimeter as associated with the number of sides of the polygon, we think of it as associated with the polygon itself; and yet a functional dependence of two quantities is really there.

If it requires a special effort to bring in the abstract notion of function in all these cases, why do it at all? The answer is, that the student who has mastered the ideas of number and function holds the whole theory of limits in the hollow of his hand. He may meet with relations that are hard to remember at first, but not with any that are hard to understand. If not altogether simple, they are perfectly clear and definite.

Let us come back to the circle with its inscribed polygons. We remember that as the number of sides of the polygon is indefinitely increased, the ratio of the perimeter of the polygon to the diameter of the circle approaches a limit which we denote by the letter π, and which is approximately 3.14159. What do we really mean by the statement that the ratio approaches π as a limit? We mean that however good an approximation to the value of π we want, we can be sure of getting it if we only take the number of sides of the polygon large enough. Whatever limit of error we choose to set (I am using the word "limit" *here* in an entirely untechnical sense) we know that all regular inscribed polygons, having more than a certain number of sides, possess in common the property that the ratio of perimeter to diameter is equal to π within the limit of error named. The assertion that a limit is approached is an abbreviation for what I have said at length, for the statement that a certain property is possessed by all regular inscribed polygons with more than a certain number of sides. In the language of a few minutes ago, it means that when the values of the ratio of perimeter to diameter are written down for the various values of n, all the values of the function corresponding to sufficiently large values of n satisfy a certain numerical condition. *It does not mean anything more than this.* The statement can be made more precise than I have made it, but not more comprehensive. It does not mean, for example, that the circle itself is a polygon of infinitely many sides, each of zero length, and that if you add up all these zeros patiently enough, one by one, you will get 3.14159 in some transcendental and supernatural way. The mathematician does not make progress by adding together infinitely many zeros, and as the president of this association has said to me more than once, how can that have infinitely many sides which has not even one?

For another illustration, consider the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots.$$

We say that the sum of the series is 2. Let us inquire how this is to be interpreted. We see that as we take more and more terms of the series, we get sums that are more and more nearly equal to 2. The sum of the first $n + 1$ terms is $2 - \frac{1}{2^n}$, a function of n which differs very little from 2 if n is large. The fact that it is never exactly equal to 2, is of secondary importance. The essential thing is, that if any limit of error is assigned, the value of the sum will differ from 2 by less than the amount specified, for all values of n that are sufficiently large. The function of n with which we are concerned is such that for all values of n beyond a certain point, the corresponding values of the function satisfy a certain numerical requirement. We express this by saying that the sum of the first $n+1$ terms of the series approaches 2 as a limit when n becomes infinite, or, more briefly still, that the series has 2 for its sum. But this does not mean anything more than that the function of n which represents the sum of the first $n+1$ terms has the property already described. It does not mean that by adding together *all* the terms of the series you get the value 2 exactly I do not object if someone more gifted with poetic imagination than I, and not a student in any of my classes, chooses to refresh himself with such a vision, but I do object with feeling to being told that that is what I mean, or any mathematician with whose ways of thought I am familiar.

I have just one more example to propound. One of the simplest functions of a variable x is the function x^2. Let us consider not merely the associated values of x and x^2 themselves, but the relation between the difference of two values of x and the difference of the corresponding values of x^2. In particular, we will let $x=1$ be one of the values of x, so that we are concerned with the relative magnitudes of the quantities $x-1$ and x^2-1^2. We shall be interested primarily in the relation between these differences when the differences themselves are very small, in other words, when x is very nearly equal to 1. Form the ratio of the two differences; we see that

$$\frac{x^2-1}{x-1}=x+1.$$

This equation has no meaning when x is equal to 1, since it involves division by zero, and division by zero is something that the mathematician does not talk about when he is at work. For values of x that are different from 1, however, and as near to 1 as you like, the equation has a perfectly definite meaning, and shows that the ratio is more and more nearly equal to 2 as the value 1 for x is approached. More specifically, if any limit of error is assigned, the value of the ratio will surely be equal to 2 within this limit of error, if only the difference between x and

1 is sufficiently small. The ratio is a function of x; when x is sufficiently near to 1, the corresponding values of the function all satisfy a certain numerical requirement. In saying that the ratio approaches 2 as a limit, and, in the more technical language of the calculus, that 2 is the value of the derivative of the function x^2 for $x = 1$, we are saying no more than we had said before.

These examples are enough to give color to my claim that the definition of limits is a very matter-of-fact undertaking when the idea of functional dependence is once firmly established. If I have wearied you by repetition, it is because it has been my task to emphasize the fact that the same ideas do appear again and again, in one connection after another. I know perfectly well that nothing that I have said is new, that it has all been better expressed many times before, but I have felt that there is nothing that I could say with more sincerity, in this hour that you have been so kind as to give me, than that I look forward to a time when the belief will be at least tolerated, if not generally accepted, that a mathematician is an honest man who walks on the earth in the light of day.

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